Three-dimensional spinning solitons in the cubic-quintic nonlinear medium

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We find one-parameter families of three-dimensional spatiotemporal bright vortex solitons (doughnuts, or *spinning light bullets*), in bulk dispersive cubic-quintic optically nonlinear media. The spinning solitons display a symmetry-breaking azimuthal instability, which leads to breakup of the spinning soliton into a set of fragments, each being a stable nonspinning light bullet. However, in some cases the instability is developing so slowly that the spinning light bullets may be regarded as virtually stable ones, from the standpoint of an experiment with finite-size samples.

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The existence of stable optical spatiotemporal solitons, or *light bullets* [1–3] (LBs), is now a well-established fact. Although LBs cannot be stable in the Kerr ($\chi^{(3)}$) medium because of the collapse [4], stability can be easily achieved in saturable [5,6] or quadratically nonlinear ($\chi^{(2)}$) media [7,8]. Although a fully localized LB in three dimensions (3D) has not yet been observed in an experiment, the first successful experiments with quasi-two-dimensional (quasi-2D) bullets in a bulk $\chi^{(2)}$ medium have been recently reported [9].

A natural and fairly interesting generalization of LB is a concept of a spinning LB in the form of a ''doughnut'' (i.e., with a hole in the middle). Here, however, the stability is a major issue, as, unlike their zero-spin counterparts, the spinning bullets are prone to be unstable against azimuthal perturbations. In the 2D models with the $\chi^{(2)}$ and saturable nonlinearities, direct numerical simulations have revealed a very strong azimuthal instability [10], which was later observed experimentally in a $\chi^{(2)}$ medium [11]. As a result of this instability, an initial LB with spin ''1'' (see an exact definition below) does not decay into radiation but instead splits into three (in the $\chi^{(2)}$ medium) or two (in the saturable one) fragments, each being a moving zero-spin bullet, so that the spin momentum is transformed into the orbital momentum of the fragments.

The first direct numerical results for a 3D spinning bullet in the $\chi^{(2)}$ model have been very recently obtained in Ref. [12]. The main result is that this bullet is strongly unstable, splitting into a set of the moving zero-spin solitons, quite similar to what is known in the 2D case. Thus, it is a challenging problem to find a physically meaningful model in which a *spinning* LB would be either stable or, at least, sufficiently weakly unstable.

A model that has a chance to feature this property is the one with a cubic-quintic (CQ) nonlinearity, which postulates a nonlinear correction to the medium's refractive index in the form $\delta n = n_2 I - n_4 I^2$, I being the light intensity. Obviously, this may be formally obtained by an expansion of the saturable nonlinearity, with $\delta n = n_2 I[1+(n_4/n_2)I]^{-1}$. However, an important difference is that while the latter nonlinearity is always self-focusing, $d(\delta n)/dI > 0$, the CQ model changes the sign of the focusing at a critical intensity $I_c = n_2/(2n_4)$. Note that an experimental measurement of the nonlinear dielectric response in the polydiacetylene *para*-toluene sulfonate (PTS) optical crystal suggests that the CQ nonlinearity, rather than the saturable one, may correctly model this medium [13].

Direct numerical simulations of the dynamics of 2D solitons with spin ''1'' in the CQ model were first reported in Ref. [14]. According to that work, the spinning 2D bullet was fairly robust provided that its energy was not too small. It has proved robust not only against small perturbations, but also against collisions, which were found to be nearly elastic. A 3D light bullet in the same model was considered very recently in [15]. However, direct stability simulations were not performed in the latter work; in particular, the stability analysis was limited to the application of the well-known Vakhitov-Kolokolov criterion [16], which is a necessary condition for the full stability of the soliton, but does not take into regard the most detrimental azimuthal perturbations.

The objective of the present work is a direct numerical analysis of the 3D spinning LBs in the CQ model, including a full study of their robustness upon propagation. The main result is that the spinning bullets are always unstable; however, their instability may be developing, depending on the LB's energy, much slower than in the $\chi^{(2)}$ model, and in some cases it is found to be so slow that the spinning bullet becomes virtually stable, from the standpoint of a possible experiment in finite-size 3D samples.

The equation governing the evolution of the electromagnetic field envelope A $(I=|A|^2)$ in a CQ isotropic dispersive medium is the cubic-quintic nonlinear Schrödinger equation,

$$2i\kappa_0 \frac{\partial A}{\partial z} + \nabla_\perp^2 A + \kappa_0 D \frac{\partial^2 A}{\partial \tau^2} + 2\kappa_0^2 (n_2/n_0)|A|^2 A - 2\kappa_0^2 (n_4/n_0)|A|^4 A = 0, \tag{1}$$

where κ_0 is the propagation constant, $D = -d^2\kappa_0/d\omega^2 > 0$ is the coefficient of temporal dispersion which is assumed *anomalous* (there is no chance to have solitons if the dispersion is normal, with $D<0$), $\tau \equiv t - z/v_g$, v_g being the group velocity of the carrier wave, is the ''reduced time,'' and the

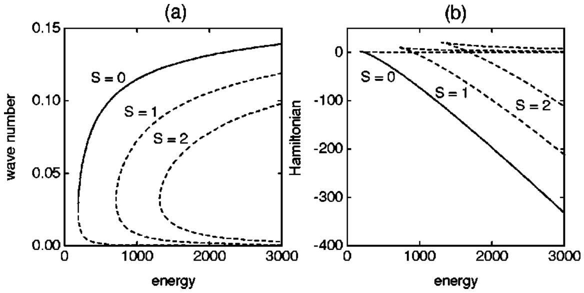

FIG. 1. The propagation constant κ (a) and Hamiltonian H (b) of the spinning light bullet vs the energy E for the doughnutlike light bullets.

Laplacian $\nabla_\perp^2$ (representing the spatial diffraction) acts on the transverse coordinates. Defining rescaled variables $u = A\sqrt{n_4/n_2}$, $T = \tau n_2\sqrt{2\kappa_0/Dn_0n_4}$, $Z = z\kappa_0 n_2^2/n_0n_4$, and $(X,Y) = (x,y)\kappa_0 n_2\sqrt{2/n_0n_4}$, one transforms Eq. (1) into a normalized form,

$$i\frac{\partial u}{\partial Z} + \left(\frac{\partial^2 u}{\partial X^2} + \frac{\partial^2 u}{\partial Y^2} + \frac{\partial^2 u}{\partial T^2}\right) + |u|^2 u - |u|^4 u = 0. \tag{2}$$

Here, X, Y, and T are, respectively, the normalized transverse spatial and temporal variables, Z being the normalized propagation distance.

First we look for stationary solutions to Eq. (2) of the form $u = U(r,T)\exp(is\theta)\exp(i\kappa Z)$, where r and θ are the polar coordinates in the transverse plane, κ is a propagation constant, and the integer s is the ''spin.'' The amplitude U can be taken real and obeys the equation

$$\left(\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} - \frac{s^2}{r^2}U + \frac{\partial^2 U}{\partial T^2}\right) - \kappa U + U^3 - U^5 = 0, \tag{3}$$

κ parametrizing the family of stationary solutions.

Equation (2) has a well-known conserved quantity (dynamical invariant), which is usually called energy (or the number of quanta) and has the form of an integrated intensity of the field,

$$E = \int\int\int |u(X,Y,T)|^2 dX dY dT. \tag{4}$$

The other obvious dynamical invariants of Eq. (2) are the Hamiltonian

$$H = \int\int\int [|u_X|^2 + |u_Y|^2 + |u_T|^2 - (1/2)|u|^4 + (1/3)|u|^6] dX dY dT, \tag{5}$$

the momentum (equal to zero for the solution considered), and the angular momentum in the transverse plane

$$L = \int\int\int (\partial\phi/\partial\theta)|u|^2 dX dY dT, \tag{6}$$

ϕ being the phase of the complex field u.

One readily finds from Eq. (3) that the values of L and H for the stationary spinning LBs are related to its energy as follows: $L = sE$ and

$$H = \kappa E - \tfrac{2}{3}\int\int 2\pi r U^6(r,T) dr dT. \tag{7}$$

We have numerically found one-parameter families of the 3D spinning-LB solutions that have the form of a doughnut with a hole in the center. In accord with the results predicted by means of the semianalytical variational approximation developed in Ref. [15], the solutions exist provided that their energy exceeds a certain threshold value. As an additional test of the accuracy of numerical computations, we have used a relationship which can be obtained directly from Eq. (3):

$$\kappa E = (1/4)\int\int 2\pi r U^4 dr dT. \tag{8}$$

To quantify the LB solutions, in Fig. 1 we show the propagation constant κ and the Hamiltonian H of the zero-spin $(s=0)$ and spinning LBs, with $s=1$ and $s=2$, vs their energy E. It is evident that, in accord with Ref. [15], the threshold energy strongly increases with the value of the spin.

In Fig. 1 full and dashed lines correspond, respectively, to stable and unstable branches, as per direct numerical results

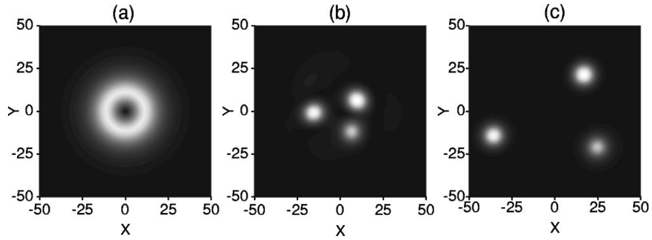

FIG. 3. Gray-scale contour plots illustrating the instability of the light bullet with spin $s=1$ and $\kappa=0.01$: (a) $Z=0$, (b) $Z=520$, and (c) $Z=600$.

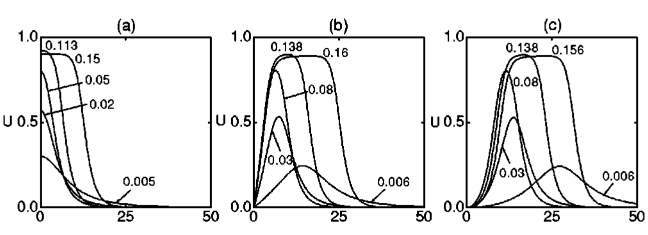

FIG. 2. The stationary spinning-bullet solutions: (a) $s=0$, (b) $s=1$, and (c) $s=2$. The values of κ are indicated near the curves.

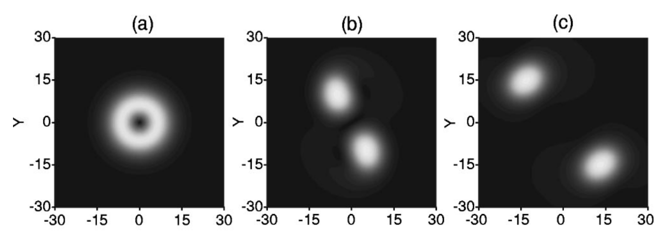

FIG. 4. The same as in Fig. 3, but for $\kappa=0.08$: (a) $Z=0$, (b) $Z=600$, and (c) $Z=630$.

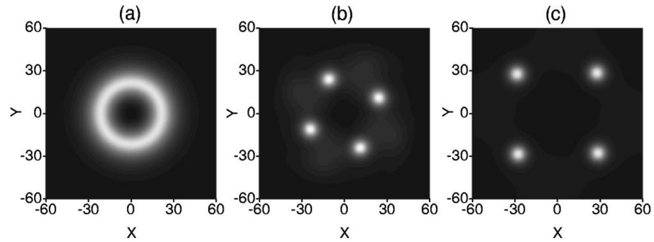

FIG. 5. Gray-scale contour plots illustrating the instability of the light bullets with spin $s=2$ and $\kappa=0.01$: (a) $Z=0$, (b) $Z=220$, and (c) $Z=320$.

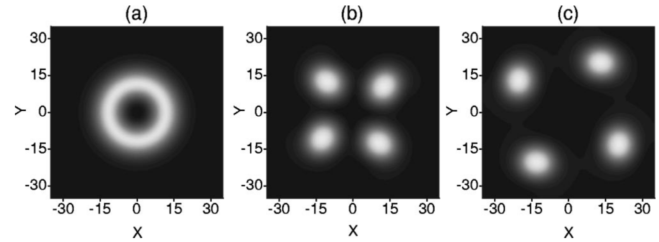

FIG. 6. The same as in Fig. 5, but for $\kappa=0.08$: (a) $Z=0$, (b) $Z=230$, and (c) $Z=260$.

for the stability presented below. In particular, it will be seen that only the $s=0$ solitons may be fully stable. The $s=0$ branch of the solutions in Fig. 1 is divided into stable and unstable portions on the basis of the above-mentioned Vakhitov-Kolokolov criterion [16].

Figure 2 shows the radial profile of the 3D solitons for several values of κ. It is seen that, with the increase of κ, the amplitude of the solitons attains a maximum at $\kappa_{\max} \simeq 0.113$ for $s=0$, and $\kappa_{\max}\simeq 0.138$ for $s=1$ and $s=2$. For $\kappa > \kappa_{\max}$, the soliton's amplitude decreases and its shape becomes flatter. It is relevant to mention that the semianalytical approximation of Ref. [15] yields very close results for the characteristics of the stationary solutions. For instance, it predicted that, for $s=1$, the threshold energy was $E_{\mathrm{thr}} = 750$, and this value was attained at $\kappa=0.033$, cf. Fig. 1(a).

Figures 3–6 show some representative gray-scale contour plots of the intensity $|u(X,Y,T=0)|^2$. They display the most important result: by direct numerical simulations of Eq. (2), we have found that the spinning LBs are *always* unstable against azimuthal perturbations. Eventually, the instability leads to a breakup of the doughnuts into several moving zero-spin solitons. Examples for $s=1$ are displayed in Figs. 3 and 4 for two different values of κ. Remarkably, the three fragments emerging in the case of a smaller initial energy of the spinning LB (smaller κ) have *unequal energies* and, accordingly, unequal intensities at their central points (Fig. 3). With the increase of the initial energy, the number of the fragments decreases from three to two. In the latter case, the two fragments have exactly equal energies (Fig. 4). For an initial LB with $s=2$, we have found that a typical outcome is splitting into four fragments with *equal energies* (see Figs. 5 and 6).

Thus the initial internal angular momentum of the spinning LB is converted into the orbital momentum of the emerging fragments, which fly out tangentially to a circular crest of the doughnut soliton. Running much more simulations, we have found that the number of the fragments is, roughly, twice the original spin *s*.

Despite the instability of the spinning LBs, they may have a chance to be observed in a finite-size bulk sample if the instability is developing slowly enough, so that the LB may be formed and survive over several LBs *soliton periods*. In terms of the propagation constant, the soliton period is $Z_0 \equiv \pi/2\kappa$. To estimate a relation between Z_0 and the propagation distance over which the LB survives, we notice that, in the case $s=1$ and $\kappa=0.01$, which corresponds to the case displayed in Fig. 3, numerical data show that the splitting actually commences at $Z_{\mathrm{split}}\simeq 400$, while the corresponding soliton's period is $Z_0 \approx 160$. The ratio $Z_{\mathrm{split}}/Z_0 \approx 2.5$ is at a border of a range in which the LB can be a physically meaningful object. However, in the case $s=1$ and $\kappa=0.08$, which corresponds to Fig. 4, we find $Z_{\mathrm{split}}\simeq 550$, while $Z_0 \approx 20$, so that in this case LB may be regarded as a *virtually stable* object. Generally, we observed a clear trend to stabilization of the spinning LBs with an increase of their energy (i.e., increase of κ). Note that, on the contrary to the situation with the CQ model, the instability of the spinning LBs in the 3D $\chi^{(2)}$ model is very strong, so that the spinning $\chi^{(2)}$ bullets cannot exist stably even over one soliton's period [12].

The instability turns out to be much stronger for $s=2$: in the case $\kappa=0.01$, we find $Z_{\mathrm{split}}\simeq 130$, and for $\kappa=0.08$, the result is $Z_{\mathrm{split}}\simeq 180$. Thus, it would be less feasible, but not impossible, to experimentally observe LBs with $s=2$. Probably, there is no chance to observe them with $s>2$.

These results, showing that all the 3D bullets in the present model are eventually unstable, calls to revisit the 2D version of the same model. Accurate simulations reveal that the 2D bullets are also subject to a relatively weak azimuthal instability (the same result was earlier obtained by Neshev [17]). The stability of the 2D bullets reported in Ref. [14] was, most probably, a result of insufficiently long simulations.

In conclusion, in the framework of the standard cubic-quintic nonlinear Schrödinger model in a 3D dispersive medium, we have found one-parameter families of spatiotemporal spinning bright solitons in the form of a doughnut with a hole in its center. All these spinning spatiotemporal solitons are subject to an instability against azimuthal perturbations, which leads to the splitting of the doughnut ''bullet'' into several fragments, each being a stable moving zero-spin soliton. However, in certain cases the splitting distance may be much larger than the corresponding soliton period, which renders experimental observation of the spinning bullets quite feasible.

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